Effects of a Logistic Reaction to Finite Difference Numerical Solutions of the Inviscid Burgers Equation

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Abstrak. Makalah ini menyelesaikan persamaan *Burgers inviscid* yang melibatkan reaksi logistik. Tujuannya adalah memeriksa pengaruh reaksi logistik pada penyelesaian numeris persamaan Burgers inviscid. Metode yang digunakan adalah beda hingga standar dan beda hingga nonstandar. Berdasarkan hasil penelitian ini, reaksi logistik pada persamaan *Burgers inviscid* membuat perambatan gelombang kejut menjadi semakin cepat.

Kata kunci: metode beda hingga; persamaan *Burgers*; reaksi logistik

Abstract. This study solves the inviscid Burgers equation involving a logistic reaction. The goal is to investigate the effects of the logistic reaction to numerical solutions of the inviscid Burgers equation. The methods used in this study were standard and nonstandard finite difference methods. Based on the results of the study, the logistic reaction in the inviscid Burgers equation makes the shock wave propagation faster.

Keywords: finite difference method; Burgers equation; logistic reaction

1. Introduction

The Burgers equation is a partial differential equation of the conservation laws. It has interesting properties of nonlinear advection. When there is no source terms, it is a hyperbolic problem. It is the simplest nonlinear representation of the Novier-Stokes equations. Therefore, studying the solution of the Burgers equation will definitely be helpful to solve the Novier-Stokes equations.

Several numerical methods are available in the literature for solving differential equations [1]. An old method is the standard finite difference method. This method is applicable for wide range of problems [2], but the accuracy sometimes needs attention. To deal with this accuracy problem, Mickens [3-6] proposed the so called nonstandard finite difference method. The method is constructed in such a way that the point-wise approximation is defined locally. In other words, the approximations at the considered nodes are actually exact solutions at those nodes. By doing so, the method leads to either exact solution or a very accurate approximate solutions.

The rest of this paper is organised as follows. Section 2 discusses the inviscid Burgers equation with a logistic reaction. Section 3 provides the numerical approach of the finite difference methods in solving the Burgers equation. Results and discussion are given in Section 4. The conclusion of the paper is in Section 5.

2. Mathematical model

The mathematical model of the inviscid Burgers equation involving a logistic reaction is [5],

$$q_t + \left(\frac{q^2}{2} \right)_x = q(1-q) \tag{1}$$

here t represents the free variable of time, x denotes the free variable of space, $q(x,t)$ is the dependent variable which represents a conserved quantity. Here $f(q) = q^2/2$ is the flux function. Equation (1) can be decomposed into three equations,

$$q_t = q(1-q), \tag{2}$$

$$q_x = 1-q, \tag{3}$$

$$q_t + qq_x = 0. \tag{4}$$

These equations (2)-(4) are used to construct a nonstandard finite difference method in order to solve equation (1).

3. Finite difference methods

Following the work of Mickens [5], the exact finite difference schemes for equations (2)-(3) are respectively given by

$$\frac{q^{k+1} - q^k}{e^{\Delta t} - 1} = q^k - q^{k+1} q^k, \tag{5}$$

$$\frac{q_{m+1} - q_m}{1 - e^{-\Delta x}} = 1 - q_m. \tag{6}$$

Here Δt is the time step, Δx is the space step, and e is the Euler's number. In addition, $q^k = q(\cdot, t^k)$, $q_m = (x_m, \cdot)$, $q_m^k = q(x_m, t^k)$. For some initials t^0 and x_0, $t^k = t^0 + k\Delta t$ and $x_m = x_0 + m\Delta x,$ where $k = 0,1,2,3,\ldots$ and $m = 0,1,2,3,\ldots$.
No exact finite difference scheme is known for solving equation (4), so the finite difference scheme is used for solving equation (1), either

$$\frac{q_m^{k+1} - q_m^k}{\Delta t} + q_m^{k+1} \left(\frac{q_m^k - q_{m-1}^k}{\Delta x} \right) = 0, \tag{7}$$

or

$$\frac{q_m^{k+1} - q_m^k}{\Delta t} + q_m^k \left(\frac{q_m^{k+1} - q_{m-1}^{k+1}}{\Delta x} \right) = 0. \tag{8}$$

However, equations (7) and (8) are nonconservative. Using schemes (5)-(6) but with a conservative flux, it is obtained the nonstandard finite difference scheme for solving equation (1) as follows

$$\frac{q_m^{k+1} - q_m^k}{e^{\Delta t} - 1} + \frac{(q_m^k)^2 - (q_{m-1}^k)^2}{2(1 - e^{-\Delta x})} = \frac{q_m^k + q_{m-1}^k}{2} (1 - q_{m-1}^{k+1}). \tag{9}$$

As a note, the standard finite difference scheme with a conservative flux is

$$\frac{q_m^{k+1} - q_m^k}{\Delta t} + \frac{(q_m^k)^2 - (q_{m-1}^k)^2}{2\Delta x} = q_m^k (1 - q_m^k). \tag{10}$$

Both standard and nonstandard finite difference methods shall be used to investigate the effects of the logistic reaction to the numerical solution of the inviscid Burgers equation.

4. Results and discussion

All quantities are assumed to be measured in SI units of the MKS system. It is considered that equation (1) in the space domain:

$$0 \le x \le 100 \tag{11}$$

where the initial condition is:

$$q(x,0) = \begin{cases} 1 & \text{if} \quad 0 \le x \le 1, \\ 0 & \text{if} \quad \text{otherwise.} \end{cases} \tag{12}$$

For numerical simulations, the boundary conditions are:

$$q(0,t) = 1 \quad \text{and} \quad q(100,t) = 0. \tag{13}$$

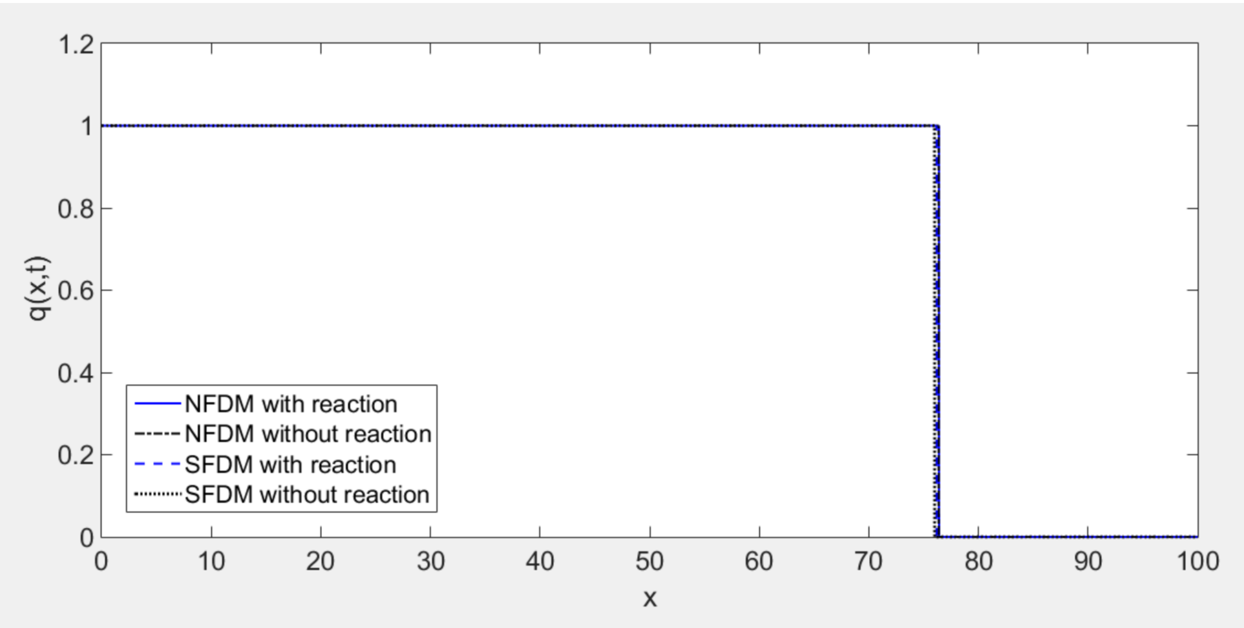

Figure 1. Simulation results of the inviscid Burgers equation with and without reaction at time (t = 150)

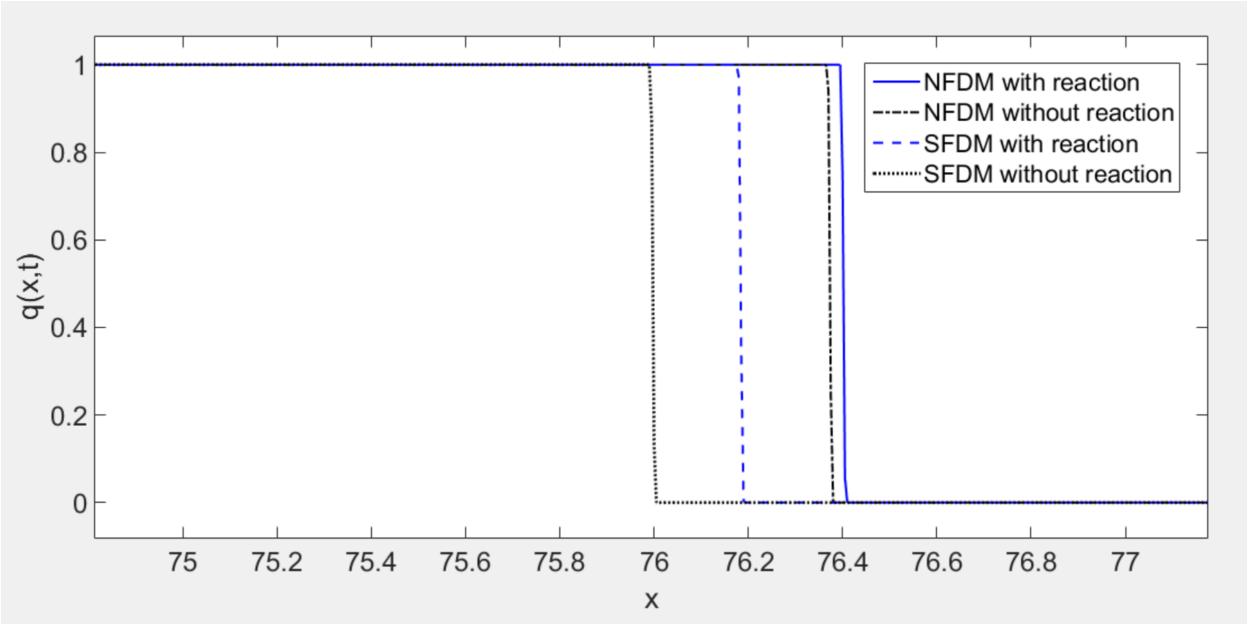

Figure 2. Magnification of the view of simulation results of the inviscid Burgers equation at time ($t = 150$)

Taking $\Delta t = \Delta x = 0.005$, the results at time $t = 150$ are shown in Figure 1 and Figure 2. Figure 1 shows simulation results of the inviscid Burgers equation at time $t = 150$ produced by the Nonstandard Finite Difference Method (NFDM) and the Standard Finite Difference Method (SFDM). To observe clearly the behaviour of the numerical solutions, this figure is magnified into Figure 2 showing the results around the shock. These figures include simulation results that involve the logistic reaction term and simulation results that do not involve the logistic reaction term.

It is obtained that the logistic reaction makes the shock wave propagate faster. This is true for both schemes, NFDM and SFDM. All numerical solutions have discrepancies one another in terms of the shock position. However, the discrepancies are relatively small in comparison to the whole spatial domain, that is, 0.4 out of 100. In other words, the discrepancies are less than 0.4%. This is shown in Figure 2, as the fastest shock has the position at about 76.4 and the slowest one at about 76.

In addition, it is also obtained that the shock wave produced by the nonstandard finite difference method propagates faster than the shock wave produced by the standard finite difference method. The reason why NFDM produces faster shock wave could be investigated further in the future.

5. Conclusion

Standard and nonstandard finite difference methods have been presented for solving the inviscid Burgers equation involving a logistic reaction. The methods solve the problem with relatively small discrepancies. It is obtained that the logistic reaction speeds up the shock wave propagation. In addition, the shock wave of the nonstandard finite difference solution moves faster than that of the standard finite difference solution. As a future research direction, this investigation about the effects of the logistic reaction could be extended to the viscous Burgers equation. Another direction is to investigate why the shock wave produced by the nonstandard finite difference method propagates faster than the shock wave produced by the standard finite difference method.

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